Adaptive Filtering - Theory and Applications

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Introduction
Estimation Techniques

Several techniques to solve estimation problems.

- **Classical Estimation**
  Maximum Likelihood (ML), Least Squares (LS), Moments, etc.

- **Bayesian Estimation**
  Minimum MSE (MMSE), Maximum A Posteriori (MAP), etc.

**Linear Estimation**

Frequently used in practice when there is a limitation in computational complexity – *Real-time operation*
Linear Estimators

Simpler to determine: depend on the first two moments of data

- **Statistical Approach** – Optimal Linear Filters
  - Minimum Mean Square Error
  - Require second order statistics of signals

- **Deterministic Approach** – Least Squares Estimators
  - Minimum Least Squares Error
  - Require handling of a data observation matrix
Limitations of Optimal Filters and LS Estimators

- Statistics of signals may not be available or cannot be accurately estimated
- There may not be available time for statistical estimation (real-time)
- Signals and systems may be non-stationary
- Memory required may be prohibitive
- Computational load may be prohibitive
Iterative Solutions

- Search the optimal solution starting from an initial guess
- **Iterative algorithms are based on classical optimization algorithms**
- Require reduced computational effort per iteration
- Need several iterations to converge to the optimal solution
- **These methods form the basis for the development of adaptive algorithms**
- Still require the knowledge of signal statistics
Adaptive Filters

Usually approximate iterative algorithms and:

- Do not require previous knowledge of the signal statistics
- Have a small computational complexity per iteration
- Converge to a neighborhood of the optimal solution

Adaptive filters are good for:

- Real-time applications, when there is no time for statistical estimation
- Applications with nonstationary signals and/or systems
Properties of Adaptive Filters

- They can operate satisfactorily in unknown and possibly time-varying environments without user intervention.
- They improve their performance during operation by learning statistical characteristics from current signal observations.
- They can track variations in the *signal operating environment* (SOE).
Adaptive Filtering Applications
Basic Classes of Adaptive Filtering Applications

- System Identification
- Inverse System Modeling
- Signal Prediction
- Interference Cancellation
System Identification

\[ x(n) \rightarrow \text{unknown system} \rightarrow y(n) + d(n) + e(n) \]

\[ e_o(n) \leftarrow - \hat{d}(n) \rightarrow \text{adaptive algorithm} \rightarrow \text{other signals} \]
Applications – System Identification

Channel Estimation
- Communications systems
- Objective: model the channel to design distortion compensation
- $x(n)$: training sequence

Plant Identification
- Control systems
- Objective: model the plant to design a compensator
- $x(n)$: training sequence
Echo Cancellation

- Telephone systems and VoIP
- Echo caused by network impedance mismatches or acoustic environment
- Objective: model the echo path impulse response
- $x(n)$: transmitted signal
- $d(n)$: echo + noise

Figure: Network Echo Cancellation
Inverse System Modeling

- Adaptive filter attempts to estimate unknown system’s inverse
- Adaptive filter input usually corrupted by noise
- Desired response $d(n)$ may not be available
Applications – Inverse System Modeling
Channel Equalization

Objective: reduce intersymbol interference
Initially – training sequence in $d(n)$
After training: $d(n)$ generated from previous decisions
Signal Prediction

- most widely used case – forward prediction
- signal $x(n)$ to be predicted from samples
  \[ \{x(n - n_o), x(n - n_o - 1), \ldots, x(n - n_o - L)\} \]
Application – Signal Prediction

DPCM Speech Quantizer - Linear Predictive Coding

- Objective: Reduce speech transmission bandwidth
- Signal transmitted all the time: quantization error
- Predictor coefficients are transmitted at low rate

\[
\begin{align*}
    d(n) + e(n) &\rightarrow \text{Quantizer} \\
    y(n) &\rightarrow \text{Predictor} \\
    y(n) + Q[e(n)] &\approx d(n)
\end{align*}
\]
Interference Cancelation

- One or more sensor signals are used to remove interference and noise
- Reference signals correlated with the interference should also be available
- Applications:
  - array processing for radar and communications
  - biomedical sensing systems
  - active noise control systems
Application – Interference Cancelation
Active Noise Control


- Cancelation of acoustic noise using destructive interference
- Secondary system between the adaptive filter and the cancelation point is unavoidable
- Cancelation is performed in the acoustic environment
Active Noise Control – Block Diagram

Adaptive Algorithm

\[ x(n) \xrightarrow{\mathbf{w}^o} d(n) + e(n) \]

\[ \mathbf{w}(n) \xrightarrow{\hat{S}} y(n) \xrightarrow{S} y_s(n) \xrightarrow{g(y_s)} y_g(n) \]

\[ z(n) \]

\[ x_f(n) \xrightarrow{\text{Adaptive Algorithm}} \hat{S} \]

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Adaptive Filtering Principles
Adaptive Filter Features

Adaptive filters are composed of three basic modules:

- **Filtering structure**
  - Determines the output of the filter given its input samples
  - Its weights are periodically adjusted by the adaptive algorithm
  - Can be linear or nonlinear, depending on the application
  - Linear filters can be FIR or IIR

- **Performance criterion**
  - Defined according to application and mathematical tractability
  - Is used to derive the adaptive algorithm
  - Its value at each iteration affects the adaptive weight updates

- **Adaptive algorithm**
  - Uses the performance criterion value and the current signals
  - Modifies the adaptive weights to improve performance
  - Its form and complexity are function of the structure and of the performance criterion
Signal Operating Environment (SOE)

Comprises all informations regarding the properties of the signals and systems

- Input signals
- Desired signal
- Unknown systems

If the SOE is nonstationary

- Acquisition or convergence mode: from start until close to best performance
- Tracking mode: readjustment following SOE’s time variations

Adaptation can be

- Supervised – desired signal is available \( e(n) \) can be evaluated
- Unsupervised – desired signal is unavailable
Performance Evaluation

- Convergence rate
- Misadjustment
- Tracking
- Robustness (disturbances and numerical)
- Computational requirements (operations and memory)
- Structure
  - facility of implementation
  - performance surface
  - stability
Optimum versus Adaptive Filters in Linear Estimation

Conditions for this study

- Stationary SOE
- Filter structure is transversal FIR
- All signals are real valued
- Performance criterion: Mean-square error $E[e^2(n)]$

The Linear Estimation Problem

$$J_{ms} = E[e^2(n)]$$
The Linear Estimation Problem

\[ x(n) = [x(n), x(n-1), \cdots, x(n-N+1)]^T \]
\[ y(n) = x^T(n)w \]
\[ e(n) = d(n) - y(n) = d(n) - x^T(n)w \]
\[ J_{ms} = E[e^2(n)] = \sigma_d^2 - 2p^T w + w^T R_{xx} w \]

where
\[ p = E[x(n)d(n)]; \quad R_{xx} = E[x(n)x^T(n)] \]

Normal Equations
\[ R_{xx} w^o = p \Leftrightarrow w^o = R_{xx}^{-1} p \quad \text{for } R_{xx} > 0 \]
\[ J_{ms_{\text{min}}} = \sigma_d^2 - p^T R_{xx}^{-1} p \]
What if $d(n)$ is nonstationary?

\[ x(n) = [x(n), x(n-1), \cdots, x(n-N+1)]^T \]
\[ y(n) = x^T(n)w(n) \]
\[ e(n) = d(n) - y(n) = d(n) - x^T(n)w(n) \]
\[ J_{ms}(n) = E[e^2(n)] = \sigma_d^2(n) - 2p(n)^T w(n) + w^T(n)R_{xx}w(n) \]

where
\[ p(n) = E[x(n)d(n)]; \quad R_{xx} = E[x(n)x^T(n)] \]

Normal Equations

\[ R_{xx}w^o(n) = p(n) \quad \Rightarrow \quad w^o(n) = R_{xx}^{-1}p(n) \quad \text{for } R_{xx} > 0 \]
\[ J_{ms_{\min}}(n) = \sigma_d^2(n) - p^T(n)R_{xx}^{-1}p(n) \]
Optimum Filters versus Adaptive Filters

Optimum Filters
- Compute
  \[ p(n) = E[x(n)d(n)] \]
- Solve \( R_{xx}w^o = p(n) \)
- Filter with \( w^o(n) \) \(\xrightarrow{\text{filter}}\)
  \[ y(n) = x^T(n)w^o(n) \]

Nonstationary SOE:
Optimum filter determined for each value of \( n \)

Adaptive Filters
- Filtering: \( y(n) = x^T(n)w(n) \)
- Evaluate error: \( e(n) = d(n) - y(n) \)
- Adaptive algorithm:
  \[ w(n + 1) = w(n) + \Delta w[x(n), e(n)] \]

\( \Delta w(n) \) is chosen so that \( w(n) \) is close to \( w^o(n) \) for \( n \) large
Characteristics of Adaptive Filters

- Search for the optimum solution on the performance surface
- Follow principles of optimization techniques
- Implement a recursive optimization solution
- Convergence speed may depend on initialization
- Have stability regions
- Steady-state solution fluctuates about the optimum
- Can track time varying SOEs better than optimum filters
- Performance depends on the performance surface
Iterative Solutions for the Optimum Filtering Problem
Performance (Cost) Functions

- **Mean-square error** \( E[e^2(n)] \) (Most popular)
  - **Adaptive algorithms:** Least-Mean Square (LMS), Normalized LMS (NLMS), Affine Projection (AP), Recursive Least Squares (RLS), etc.

- **Regularized MSE**
  \[
  J_{rms} = E[e^2(n)] + \alpha \| \mathbf{w}(n) \|^2
  \]
  - **Adaptive algorithm:** leaky least-mean square (leaky LMS)

- **\( \ell_1 \) norm criterion**
  \[
  J_{\ell_1} = E[|e(n)|]
  \]
  - **Adaptive algorithm:** Sign-Error

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Performance (Cost) Functions – continued

- **Least-mean fourth (LMF)** criterion
  \[ J_{LMF} = E[e^4(n)] \]

  **Adaptive algorithm:** Least-Mean Fourth (LMF)

- **Least-mean-mixed-norm (LMMN)** criterion
  \[ J_{LMMN} = E[\alpha e^2(n) + \frac{1}{2}(1 - \alpha)e^4(n)] \]

  **Adaptive algorithm:** Least-Mean-Mixed-Norm (LMMN)

- **Constant-modulus** criterion
  \[ J_{CM} = E[(\gamma - |x^T(n)w(n)|^2)^2] \]

  **Adaptive algorithm:** Constant-Modulus (CM)
MSE Performance Surface – Small Input Correlation
MSE Performance Surface – Large Input Correlation
The Steepest Descent Algorithm – Stationary SOE

Cost Function

\[ J_{ms}(n) = E[e^2(n)] = \sigma_d^2 - 2p^T w(n) + w^T(n)R_{xx}w(n) \]

Weight Update Equation

\[ w(n + 1) = w(n) + \mu c(n) \]

\( \mu \): step-size
\( c(n) \): correction term (determines direction of \( \Delta w(n) \))

Steepest descent adjustment:

\[ c(n) = -\nabla J_{ms}(n) \quad \Rightarrow \quad J_{ms}(n + 1) \leq J_{ms}(n) \]

\[ w(n + 1) = w(n) + \mu [p - R_{xx}w(n)] \]
Weight Update Equation About the Optimum Weights

Weight Error Update Equation

\[ w(n + 1) = w(n) + \mu[p - R_{xx}w(n)] \]

Using \( p = R_{xx}w^o \)

\[ w(n + 1) = (I - \mu R_{xx})w(n) + \mu R_{xx}w^o \]

Weight error vector: \( v(n) = w(n) - w^o \)

\[ v(n + 1) = (I - \mu R_{xx})v(n) \]

- Matrix \( I - \mu R_{xx} \) must be stable for convergence (\(|\lambda_i| < 1\))
- Assuming convergence, \( \lim_{n \to \infty} v(n) = 0 \)
Convergence Conditions

\[ v(n + 1) = (I - \mu R_{xx})v(n); \quad R_{xx} \quad \text{positive definite} \]

**Eigen-decomposition of** \( R_{xx} \)

\[ R_{xx} = Q\Lambda Q^T \]

\[ v(n + 1) = (I - \mu Q\Lambda Q^T)v(n) \]

\[ Q^T v(n + 1) = Q^T v(n) - \mu \Lambda Q^T v(n) \]

Defining \( \tilde{v}(n + 1) = Q^T v(n + 1) \)

\[ \tilde{v}(n + 1) = (I - \mu \Lambda)\tilde{v}(n) \]
Convergence Properties

\[ \tilde{v}(n + 1) = (I - \mu \Lambda)\tilde{v}(n) \]

- \(\tilde{v}_k(n + 1) = (1 - \mu \lambda_k)\tilde{v}_k(n), \ k = 1, \ldots, N\)
- \(\tilde{v}_k(n) = (1 - \mu \lambda_k)^n\tilde{v}_k(0)\)

Convergence modes
- monotonic if \( 0 < 1 - \mu \lambda_k < 1 \)
- oscillatory if \( -1 < 1 - \mu \lambda_k < 0 \)

Convergence if \(|1 - \mu \lambda_k| < 1 \Leftrightarrow 0 < \mu < \frac{2}{\lambda_{\text{max}}}\)
Optimal Step-Size

\[ \tilde{v}_k(n) = (1 - \mu \lambda_k)^n \tilde{v}_k(0); \] convergence modes: \( 1 - \mu \lambda_k \)

- \( \max |1 - \mu \lambda_k|: \) slowest mode
- \( \min |1 - \mu \lambda_k|: \) fastest mode
Optimal Step-Size – continued

\[ |1 - \mu \lambda_k| \]

\[ \frac{1}{\lambda_{\max}} \quad \frac{1}{\lambda_{\min}} \]

\[ \min_{\mu} \max \{|1 - \mu \lambda_k|\} \]

\[ 1 - \mu^o \lambda_{\min} = -(1 - \mu^o \lambda_{\max}) \]

\[ \mu^o = \frac{2}{\lambda_{\max} + \lambda_{\min}} \]

Optimal slowest modes: \[ \pm \frac{\rho - 1}{\rho + 1}; \quad \rho = \frac{\lambda_{\max}}{\lambda_{\min}} \]
The Learning Curve – $J_{ms}(n)$

\[
J_{ms}(n) = J_{ms_{\text{min}}} + \tilde{\mathbf{v}}^T(n) \Delta \tilde{\mathbf{v}}(n) = J_{ms_{\text{min}}} + \sum_{k=1}^{N} \lambda_k \tilde{v}_k^2(n)
\]

Since $\tilde{v}_k(n) = (1 - \mu \lambda_k)^n \tilde{v}_k(0)$,

\[
J_{ms}(n) = J_{ms_{\text{min}}} + \sum_{k=1}^{N} \lambda_k (1 - \mu \lambda_k)^{2n} \tilde{v}_k^2(0)
\]

- $\lambda_k (1 - \mu \lambda_k)^2 > 0 \iff$ monotonic convergence
- Stability limit is again $0 < \mu < \frac{2}{\lambda_{\text{max}}}$
- $J_{ms}(n)$ converges faster than $w(n)$
- Algorithm converges faster as $\rho = \lambda_{\text{max}}/\lambda_{\text{min}} \to 1$
Simulation Results

\[ x(n) = \alpha x(n-1) + v(n) \]

- Input: White noise \((\rho = 1)\)
- Linear system identification - FIR with 20 coefficients
- Step-size \(\mu = 0.3\)
- Noise power \(\sigma_v^2 = 10^{-6}\)

Input: AR(1), \(\alpha = 0.7 \ (\rho = 5.7)\)
The Newton Algorithm

- Steepest descent: linear approx. of $J_{ms}$ about the operating point
- Newton’s method: Quadratic approximation of $J_{ms}$

Expanding $J_{ms}(w)$ in Taylor’s series about $w(n)$,

$$
J_{ms}(w) \sim J_{ms}[w(n)] + \nabla^T J_{ms}[w - w(n)] + \frac{1}{2}[w - w(n)]^T H(n)[w - w(n)]
$$

Differentiating w.r.t. $w$ and equating to zero at $w = w(n + 1)$,

$$
\nabla J_{ms}[w(n + 1)] = \nabla J_{ms}[w(n)] + H[w(n)][w(n + 1) - w(n)] = 0
$$

$$
w(n + 1) = w(n) - H^{-1}[w(n)]\nabla J_{ms}[w(n)]
$$
The Newton Algorithm – continued

\[ \nabla J_{ms}[w(n)] = -2p + 2R_{xx}w(n) \]
\[ H(w(n)) = 2R_{xx} \]

Thus, adding a step-size control,

\[
\begin{align*}
    w(n + 1) &= w(n) - \mu R_{xx}^{-1}[-p + R_{xx}w(n)]
\end{align*}
\]

- Quadratic surface \(\leftrightarrow\) conv. in one iteration for \(\mu = 1\)
- Requires the determination of \(R_{xx}^{-1}\)
- Can be used to derive simpler adaptive algorithms
- When \(H(n)\) is close to singular \(\leftrightarrow\) regularization

\[
\tilde{H}(n) = 2R_{xx} + 2\varepsilon I
\]
Basic Adaptive Algorithms
Least Mean Squares (LMS) Algorithm

- Can be interpreted in different ways
- Each interpretation helps understanding the algorithm behavior
- Some of these interpretations are related to the steepest descent algorithm
LMS as a Stochastic Gradient Algorithm

- Suppose we use the estimate $J_{ms}(n) = E[e^2(n)] \approx e^2(n)$
- The estimated gradient vector becomes

$$\hat{\nabla} J_{ms}(n) = \frac{\partial e^2(n)}{\partial w(n)} = 2e(n) \frac{\partial e(n)}{\partial w(n)}$$

- Since $e(n) = d(n) - x^T(n)w(n)$,

$$\hat{\nabla} J_{ms}(n) = -2e(n)x(n) \quad \text{(stochastic gradient)}$$

and, using the steepest descent weight update equation,

$$w(n + 1) = w(n) + \mu e(n)x(n) \quad \text{(LMS weight update)}$$
LMS as a Stochastic Estimation Algorithm

- \( \nabla J_{ms}(n) = -2p + 2R_{xx}w(n) \)

- Stochastic estimators
  
  \[
  \hat{p} = d(n)x(n) \quad \hat{R}_{xx} = x(n)x^T(n)
  \]

  Then,
  
  \[
  \hat{\nabla} J_{ms}(n) = -2d(n)x(n) + 2x(n)x^T(n)w(n)
  \]

- Using \( \hat{\nabla} J_{ms}(n) \) is the steepest descent weight update,
  
  \[
  w(n + 1) = w(n) + \mu e(n)x(n)
  \]
LMS – A Solution to a Local Optimization

- Error expressions
  \[ e(n) = d(n) - x^T(n)w(n) \quad (a \text{ priori error}) \]
  \[ e(n) = d(n) - x^T(n)w(n + 1) \quad (a \text{ posteriori error}) \]

- We want to maximize \(|\epsilon(n) - e(n)|\) with \(|\epsilon(n)| < |e(n)|\)

  \[ \epsilon(n) - e(n) = -x^T(n)\Delta w(n) \]
  \[ \Delta w(n) = w(n + 1) - w(n) \]

- Expressing \(\Delta w(n)\) as \(\Delta w(n) = \Delta \tilde{w}(n)e(n)\)

  \[ \epsilon(n) - e(n) = -x^T(n)\Delta \tilde{w}(n)e(n) \]

- For max \(|\epsilon(n) - e(n)| \iff \Delta \tilde{w}(n)\) in the direction of \(x(n)\)
\[ \Rightarrow \Delta \tilde{w}(n) = \mu \tilde{x}(n) \text{ and} \]

\[ \Delta w(n) = \mu \tilde{x}(n)e(n) \]

and

\[ w(n + 1) = w(n) + \mu e(n)x(n) \]

As

\[ \epsilon(n) - e(n) = -x^T(n)\Delta w(n) = -\mu x^T(n)x(n)e(n) \]

\[ |\epsilon(n)| < |e(n)| \text{ requires } |1 - \mu x^T(n)x(n)| < 1, \text{ or} \]

\[ 0 < \mu < \frac{2}{\|x(n)\|^2} \text{ (stability region)} \]
Observations - LMS Algorithm

- LMS is a noisy approximation of the steepest descent algorithm.
- The gradient estimate is unbiased.
- The errors in the gradient estimate lead to $J_{ms_{ex}}(\infty) \neq 0$.
- Vector $w(n)$ is now random.
- Steepest descent properties are no longer guaranteed.
  - $\Rightarrow$ *LMS analysis required*.
- The instantaneous estimates allow tracking without redesign.
Some Research Results


Some Research Results – continued

Some Research Results – continued

The Normalized LMS Algorithm – NLMS

- Most Employed adaptive algorithm in real-time applications
- Like LMS has different interpretations (even more)
- Alleviates a drawback of the LMS algorithm

\[ w(n + 1) = w(n) + \mu e(n)x(n) \]

- If amplitude of \( x(n) \) is large ⟷ Gradient noise amplification
- Sub-optimal performance when \( \sigma_x^2 \) varies with time (for instance, speech)
Error expressions

\[ e(n) = d(n) - x^T(n)w(n) \quad (a \text{ priori} \text{ error}) \]
\[ \epsilon(n) = d(n) - x^T(n)w(n + 1) \quad (a \text{ posteriori} \text{ error}) \]

We want to maximize \(|\epsilon(n) - e(n)|\) with \(|\epsilon(n)| < |e(n)|\)

\[ \epsilon(n) - e(n) = -x^T(n)\Delta w(n) \quad (A) \]
\[ \Delta w(n) = w(n + 1) - w(n) \]

For \(|\epsilon(n)| < |e(n)|\) we impose the restriction

\[ \epsilon(n) = (1 - \mu)e(n), \quad |1 - \mu| < 1 \]

\[ \Rightarrow \quad \epsilon(n) - e(n) = -\mu e(n) \quad (B) \]

For max \(|\epsilon(n) - e(n)|\) \Rightarrow \Delta w(n) in the direction of \(x(n)\)

\[ \Rightarrow \Delta w(n) = kx(n) \quad (C) \]
• Using (A), (B) and (C),

\[
k = \mu \frac{e(n)}{\mathbf{x}^T(n) \mathbf{x}(n)}
\]

and

\[
\mathbf{w}(n + 1) = \mathbf{w}(n) + \mu \frac{e(n) \mathbf{x}(n)}{\mathbf{x}^T(n) \mathbf{x}(n)}
\]

(NLMS weight update)
NLMS – Solution to a Constrained Optimization Problem

- Error sequence
  \[ y(n) = x^T(n)w(n) \quad \text{(estimate of } d(n)) \]
  \[ e(n) = d(n) - y(n) = d(n) - x^T(n)w(n) \quad \text{(estimation error)} \]

- Optimization (principle of minimal disturbance)
  
  Minimize \[ \|\Delta w(n)\|^2 = \|w(n + 1) - w(n)\|^2 \]
  
  subject to: \[ x^T(n)w(n + 1) = d(n) \]

  This problem can be solved using the method of Lagrange multipliers
Using Lagrange multipliers, we minimize

\[ f[w(n+1)] = \|w(n+1) - w(n)\|^2 + \lambda [d(n) - x^T(n)w(n+1)] \]

Differentiating w.r.t. \( w(n+1) \) and equating the result to zero,

\[ w(n+1) = w(n) + \frac{1}{2} \lambda x(n) \]  \hspace{1cm} (*)

Using this result in \( x^T(n)w(n+1) = d(n) \) yields

\[ \lambda = \frac{2e(n)}{x^T(n)x(n)} \]

Using this result in (*) yields

\[ w(n+1) = w(n) + \mu \frac{e(n)x(n)}{x^T(n)x(n)} \]
NLMS as an Orthogonalization Process

- **Conditions for the analysis**
  - $e(n) = d(n) - x^T(n)w(n)$
  - $w^o$ is the optimal solution (Wiener solution)
  - $d(n) = x^T(n)w^o$ (no noise)
  - $v(n) = w(n) - w^o$ (weight error vector)

- **Error signal**

  $$e(n) = d(n) - x^T(n)[v(n) + w^o]$$
  $$= x^T(n)w^o - x^T(n)[v(n) + w^o]$$
  $$= -x^T(n)v(n)$$
\[ e(n) = -\mathbf{x}^T(n)\mathbf{v}(n) \]

- **Interpretation:** To minimize the error \( \mathbf{v}(n) \) should be orthogonal to all input vectors.
- **Restriction:** We have only one vector \( \mathbf{x}(n) \).

- **Iterative solution:**
  We can subtract from \( \mathbf{v}(n) \) its component in the direction of \( \mathbf{x}(n) \) at each iteration.

- If there are \( N \) adaptive coefficients, \( \mathbf{v}(n) \) could be reduced to zero after \( N \) orthogonal input vectors.
- **Iterative projection extraction**
  \( \Rightarrow \) Gram-Schmidt orthogonalization
Recursive orthogonalization

\[ n = 0 : \quad v(1) = v(0) - \text{proj. of } v(0) \text{ onto } x(0) \]
\[ n = 1 : \quad v(2) = v(1) - \text{proj. of } v(1) \text{ onto } x(1) \]
\[ \vdots \]
\[ n + 1 : \quad v(n+1) = v(n) - \text{proj. of } v(n) \text{ onto } x(n) \]

Projection of \( v(n) \) onto \( x(n) \)

\[ P_{x(n)}[v(n)] = \left\{ x(n)[x^T(n)x(n)]^{-1}x^T(n) \right\} v(n) \]

Weight update equation

\[ v(n + 1) = v(n) - \mu \left[ x^T(n)x(n) \right]^{-1}x^T(n)v(n) x(n) - e(n)x(n) \]
\[ = v(n) + \mu \frac{e(n)x(n)}{x^T(n)x(n)} \]
NLMS has its own problems!

- NLMS solves the LMS gradient error amplification problem, but ...
- What happens if $\|x(n)\|^2$ gets too small?
- One needs to add some regularization

$$\mathbf{w}(n + 1) = \mathbf{w}(n) + \mu \frac{e(n)x(n)}{x^T(n)x(n) + \varepsilon} \quad (\varepsilon - \text{NLMS})$$
\(\varepsilon\text{-NLMS} \text{ – Stochastic Approximation of Regularized Newton}\)

- **Regularized Newton Algorithm**

\[
\mathbf{w}(n+1) = \mathbf{w}(n) + \mu[\varepsilon \mathbf{I} + \mathbf{R}_{xx}]^{-1}[\mathbf{p} - \mathbf{R}_{xx}\mathbf{w}(n)]
\]

- **Instantaneous estimates**

\[
\hat{\mathbf{p}} = \mathbf{x}(n)d(n) \\
\hat{\mathbf{R}}_{xx} = \mathbf{x}(n)\mathbf{x}^T(n)
\]

- **Using these estimates**

\[
\mathbf{w}(n+1) = \mathbf{w}(n) + \mu[\varepsilon \mathbf{I} + \mathbf{x}(n)\mathbf{x}^T(n)]^{-1}\mathbf{x}(n) \left[\underbrace{d(n) - \mathbf{x}^T(n)\mathbf{w}(n)}_{e(n)}\right]
\]

\[
\mathbf{w}(n+1) = \mathbf{w}(n) + \mu[\varepsilon \mathbf{I} + \mathbf{x}(n)\mathbf{x}^T(n)]^{-1}\mathbf{x}(n)e(n)
\]
\[ \mathbf{w}(n + 1) = \mathbf{w}(n) + \mu [\varepsilon \mathbf{I} + \mathbf{x}(n)\mathbf{x}^T(n)]^{-1} \mathbf{x}(n)e(n) \]

- Inversion of \( \varepsilon \mathbf{I} + \mathbf{x}(n)\mathbf{x}^T(n) \)?
- Matrix Inversion Formula
  \[
  [A + BCD]^{-1} = A^{-1} - A^{-1} B [C^{-1} + DA^{-1} B]^{-1} DA^{-1}
  \]
- Thus,
  \[
  [\varepsilon \mathbf{I} + \mathbf{x}(n)\mathbf{x}^T(n)]^{-1} = \varepsilon^{-1} \mathbf{I} - \frac{\varepsilon^{-2}}{1 + \varepsilon^{-1} \mathbf{x}^T(n)\mathbf{x}(n)} \mathbf{x}(n)\mathbf{x}^T(n)
  \]
- Post-multiplying both sides by \( \mathbf{x}(n) \) and rearranging
  \[
  [\varepsilon \mathbf{I} + \mathbf{x}(n)\mathbf{x}^T(n)]^{-1} \mathbf{x}(n) = \frac{\mathbf{x}(n)}{\varepsilon + \mathbf{x}^T(n)\mathbf{x}(n)}
  \]
  and
  \[
  \mathbf{w}(n + 1) = \mathbf{w}(n) + \mu \frac{e(n)\mathbf{x}(n)}{\mathbf{x}^T(n)\mathbf{x}(n) + \varepsilon}
  \]
Some Research Results


The Affine Projection Algorithm

Consider the NLMS algorithm but using 
\( \{x(n), x(n-1), \ldots, x(n-P)\} \)

Minimize 
\[ \| \Delta w(n) \|^2 = \| w(n+1) - w(n) \|^2 \]

subject to:
\[
\begin{align*}
  x^T(n)w(n+1) &= d(n) \\
  x^T(n-1)w(n+1) &= d(n-1) \\
  &\vdots \\
  x^T(n-P)w(n+1) &= d(n-P)
\end{align*}
\]
• Observation matrix

\[ X(n) = [x(n), x(n - 1), \ldots, x(n - P)] \]

• Desired signal vector

\[ d(n) = [d(n), d(n - 1), \ldots, d(n - P)] \]

• Error vector

\[ e(n) = [e(n), e(n - 1), \ldots, e(n - P)] = d(n) - X^T(n)w(n) \]

• Vector of the constraint errors

\[ e_c(n) = d(n) - X^T(n)w(n + 1) \]
Using Lagrange multipliers, we minimize

\[ f[w(n + 1)] = \|w(n + 1) - w(n)\|^2 + \lambda^T[d(n) - X^T(n)w(n + 1)] \]

Differentiating w.r.t. \(w(n + 1)\) and equating the result to zero,

\[ w(n + 1) = w(n) + \frac{1}{2}X(n)\lambda \quad (*) \]

Using this result in \(X^T(n)w(n + 1) = d(n)\) yields

\[ \lambda = 2[X^T(n)X(n)]^{-1}e(n) \]

Using this result in (*) yields

\[ w(n + 1) = w(n) + \mu X(n)[X^T(n)X(n)]^{-1}e(n) \]
Affine Projection
Solution to the Underdetermined Least-Squares Problem

We want to minimize
\[ \| e_c(n) \|^2 = \| d(n) - X^T(n)w(n + 1) \|^2 \]

where \( X^T(n) \) is \((P + 1) \times N\) with \((P + 1) < N\)

Thus, we look for the least-squares solution of the underdetermined system
\[ X^T(n)w(n + 1) = d(n) \]

The solution is
\[ w(n + 1) = [X^T(n)]^+ d(n) = X(n)[X^T(n)X(n)]^{-1}d(n) \]

Using \( d(n) = e(n) + X^T(n)w(n) \) yields
\[ w(n + 1) = w(n) + \mu X(n)[X^T(n)X(n)]^{-1}e(n) \]
Observations:

- Order of the AP algorithm: $P + 1$ ($P = 0 \rightarrow$ NLMS)
- Convergence speed increases with $P$ (but not linearly)
- Computational complexity increases with $P$ (not linearly)
- If $X^T(n)X(n)$ is close to singular, use $X^T(n)X(n) + \varepsilon I$
- The scalar error of NLMS becomes a vector error in AP (except for $\mu = 1$)
Affine Projection – Stochastic Approximation of Regularized Newton

- Regularized Newton Algorithm

\[ \mathbf{w}(n + 1) = \mathbf{w}(n) + \mu [\tilde{\epsilon} \mathbf{I} + \mathbf{R}_{xx}]^{-1} [\mathbf{p} - \mathbf{R}_{xx} \mathbf{w}(n)] \]

- Estimates using time window statistics

\[ \hat{\mathbf{p}} = \frac{1}{P + 1} \sum_{k=n-P}^{n} \mathbf{X}(k) \mathbf{d}(k) = \frac{1}{P + 1} \mathbf{X}(n) \mathbf{d}(n) \]

\[ \hat{\mathbf{R}}_{xx} = \frac{1}{P + 1} \sum_{k=n-P}^{n} \mathbf{X}(k) \mathbf{X}^T(k) = \frac{1}{P + 1} \mathbf{X}(n) \mathbf{X}^T(n) \]

- Using these estimates with \( \tilde{\epsilon} = \epsilon / (P + 1) \),

\[ \mathbf{w}(n + 1) = \mathbf{w}(n) + \mu [\epsilon \mathbf{I} + \mathbf{X}(n) \mathbf{X}^T(n)]^{-1} \mathbf{X}(n) \left[ \mathbf{d}(n) - \mathbf{X}^T(n) \mathbf{w}(n) \right] \]

\[ \mathbf{w}(n + 1) = \mathbf{w}(n) + \mu [\epsilon \mathbf{I} + \mathbf{X}(n) \mathbf{X}^T(n)]^{-1} \mathbf{X}(n) \mathbf{e}(n) \]
\[ w(n + 1) = w(n) + \mu [\varepsilon I + X(n)X^T(n)]^{-1} X(n)e(n) \]

- Using the Matrix Inversion Formula

\[ [\varepsilon I + X(n)X^T(n)]^{-1} = X(n)[\varepsilon I + X^T(n)X(n)]^{-1} \]

and

\[ w(n + 1) = w(n) + \mu X(n)[\varepsilon I + X^T(n)X(n)]^{-1} e(n) \]
Affine Projection Algorithm as a Projection onto an Affine Subspace

- Conditions for the analysis
  - $e(n) = d(n) - X^T(n)w(n)$
  - $d(n) = X^T(n)w^o$ defines the optimal solution in the least-squares sense
  - $v(n) = w(n) - w^o$  (weight error vector)

- Error vector

$$e(n) = d(n) - X^T(n)[v(n) + w(n + 1)]$$

$$= X^T(n)w(n + 1) - X^T(n)[v(n) + w(n + 1)]$$

$$= -X^T(n)v(n)$$
\[ e(n) = -X^T(n)v(n) \]

- **Interpretation:** To minimize the error \( v(n) \) should be orthogonal to all input vectors.

- **Restriction:** We are going to use only \( \{x(n), x(n - 1), \ldots, x(n - P)\} \)

- **Iterative solution:**
  We can subtract from \( v(n) \) its projection onto the range of \( X(n) \) at each iteration

  \[
  v(n + 1) = v(n) - P_{X(n)}v(n) \quad \text{(Proj. onto an affine subspace)}
  \]

- Using \( X^T(n)v(n) = -e(n) \)

  \[
  v(n + 1) = v(n) - \left\{ X(n)[X^T(n)X(n)]^{-1}X^T(n) \right\} v(n) \\
  = v(n) + \mu X(n)[X^T(n)X(n)]^{-1}e(n) \quad \text{(AP algorithm)}
  \]
Pseudo Affine Projection Algorithm

- Major problem with the AP algorithm
  For \( \mu \neq 1 \), \( e(n) \rightarrow e(n) \) \( \Rightarrow \) large computational complexity
- The Pseudo-AP algorithm replaces the input \( x(n) \) with its \( P \)-th order autoregressive prediction

\[
\hat{x}(n) = \sum_{k=1}^{P} a_k x(n - k)
\]

- Using the least-squares solution for \( a = [a_1, a_2, \ldots, a_P]^T \),

\[
a = [X_p^T(n)X_p(n)]^{-1}X_p^T(n)x(n), \quad X_p(n) = [x(n-1), \ldots, x(n-P)]
\]

- Now, we subtract from \( x(n) \) its projections onto the last \( P \) input vectors

\[
\phi(n) = x(n) - X_p(n)a(n)
\]

\[
= x(n) - \{X_p(n)[X_p^T(n)X_p(n)]^{-1}X_p^T(n)\} x(n)
\]

\[
= (I - P_P)x(n); \quad (P_P: \text{projection matrix})
\]
Example – Pseudo-AP versus AP

Mean Square Error

AR(8)=[-0.9 0.8 -0.7 0.6 -0.5 0.4 -0.3 0.2]
N=64
P=8
SNR=80dB
S=Null

α=0.7

AP(-72.82dB) PAP(-75.82)
Using $\phi$ as the new input vector for the NLMS algorithm,

$$\mathbf{w}(n + 1) = \mathbf{w}(n) + \mu \frac{\phi(n)}{\phi^T(n)\phi(n)} e(n)$$

It can be shown that Pseudo-AP is identical to AP for an AR input and $\mu = 1$

Otherwise, Pseudo-AP is different from AP

Simpler to implement than AP for $\mu \neq 1$

Can lead even to better steady-state results than AP for AR inputs

For AR inputs: NLMS with input “orthogonalization”
Some Research Results


Deterministic Algorithms
Recursive Least Squares Algorithm (RLS)

- Based on a deterministic philosophy
- Designed for the present realization of the input signals
- Least squares method adapted for real time processing of temporal series
- Convergence speed is not strongly dependent on the input statistics
RLS – Problem Definition

- Define
  
  \[\mathbf{x}_o(n) = [x(n), x(n - 1), \ldots, x(0)]^T\]
  \[\mathbf{x}(n) = [x(n), x(n - 1), \ldots, x(n - N + 1)]^T\]
  
  input to the \(N\)-tap filter

- Desired signal \(d(n)\)

- Estimator of \(d(n)\): \(y(n) = \mathbf{x}^T(n)\mathbf{w}(n)\)

- Cost function – Squared Error

  \[J_{ls}(n) = \sum_{k=0}^{n} e^2(k) = \sum_{k=0}^{n} [d(k) - \mathbf{x}^T(k)\mathbf{w}(n)]^2\]
  
  \[= \mathbf{e}^T(n)\mathbf{e}(n), \quad \mathbf{e}(n) = [e(n), e(n - 1), \ldots, e(0)]^T\]
In vector form

\[ d(n) = [d(n), d(n-1), \ldots, d(0)]^T \]
\[ y(n) = [y(n), y(n-1), \ldots, y(0)]^T, \quad y(k) = x^T(k)w(n) \]
\[ y(n) = \sum_{k=0}^{N-1} w_k(n) x_o(n-k) = \Theta(n)w(n) \]

\[ \Theta(n) = \begin{bmatrix} x(n) & x(n-1) & \cdots & x(n-N+1) \\ x(n-1) & x(n-2) & \cdots & x(n-N) \\ \vdots & \vdots & \vdots & \vdots \\ x(0) & x(-1) & \cdots & x(-N+1) \end{bmatrix} \]
\[ = [x_o(n), x_o(n-1), \ldots x_o(n-N+1)] \]

\[ e(n) = d(n) - \Theta(n)w(n) \]
\[ e(n) = d(n) - \Theta(n)w(n) \]

and we minimize

\[ J_{ls}(n) = \|e(n)\|^2 \]

- Normal Equations

\[ \Theta^T(n)\Theta(n)w(n) = \Theta^T(n)d(n) \]

\[ \tilde{R}(n)w(n) = \tilde{p}(n) \]

- Alternative representation for $\tilde{R}$ and $\tilde{p}$

\[ \tilde{R}(n) = \Theta^T(n)\Theta(n) = \sum_{k=0}^{n} x(k)x^T(k) \]

\[ \tilde{p}(n) = \Theta^T(n)d(n) = \sum_{k=0}^{n} x(k)d(k) \]
• $w(n)$ remains fixed for $0 \leq k \leq n$ to determine $J_{ls}(n)$

• Optimum vector $w(n)$ is $w^o(n) = \tilde{R}^{-1}(n)\tilde{p}$

• The algorithm has growing memory

• In an adaptive implementation,

\[ w^o(n) = \tilde{R}^{-1}(n)\tilde{p} \]

must be solved for each iteration $n$

• Problems for an adaptive implementation
  ▶ Dificulties with nonstationary signals (infinite data window)
  ▶ Computational complexity
RLS - Forgetting the Past

- Modified cost function

\[
J_{ls}(n) = \sum_{k=0}^{n} \lambda^{n-k} e^2(k) = \sum_{k=0}^{n} \lambda^{n-k} [d(k) - x^T(k)w(n)]^2
\]

\[= e^T(n)\Lambda e(n), \quad \Lambda = \text{diag}[1, \lambda, \lambda^2, \ldots, \lambda^n]\]

- Modified Normal Equations

\[
\Theta^T(n)\Lambda\Theta(n)w(n) = \Theta^T(n)\Lambda d(n)
\]

\[
\hat{R}(n)w(n) = \hat{p}(n)
\]

where

\[
\hat{R}(n) = \Theta^T(n)\Lambda\Theta(n) = \sum_{k=0}^{n} \lambda^{n-k} x(k)x^T(k)
\]

\[
\hat{p}(n) = \Theta^T(n)\Lambda d(n) = \sum_{k=0}^{n} \lambda^{n-k} x(k)d(k)
\]
RLS – Recursive Updating

- Correlation Matrix

\[
\hat{R}(n) = \sum_{k=0}^{n-1} \lambda^{n-k} x(k)x^T(k) \\
= \sum_{k=0}^{n-1} \lambda^{n-k} x(k)x^T(k) + x(n)x^T(n) \\
= \lambda \hat{R}(n-1) + x(n)x^T(n)
\]

- Cross-correlation vector

\[
\hat{p}(n) = \sum_{k=0}^{n-1} \lambda^{n-k} x(k)d(k) \\
= \sum_{k=0}^{n-1} \lambda^{n-k} x(k)d(k) + x(n)d(n) \\
= \lambda \hat{p}(n-1) + x(n)d(n)
\]
We have recursive updating expressions for $\hat{R}(n)$ and $\hat{p}(n)$.

However, we need a recursive updating for $\hat{R}^{-1}(n)$, as

$$w^o(n) = \hat{R}^{-1}(n)\hat{p}(n)$$

Applying the matrix inversion lemma to

$$\hat{R}(n) = \lambda \hat{R}(n-1) + x(n)x^T(n)$$

and defining $P(n) = \hat{R}^{-1}(n)$ yields

$$P(n) = \lambda^{-1}P(n-1) - \frac{\lambda^{-2}P(n-1)x(n)x^T(n)P(n-1)}{1 + \lambda^{-1}x^T(n)P(n-1)x(n)}$$
The Gain Vector

Definition

\[ k(n) = \frac{\lambda^{-1} P(n - 1) x(n)}{1 + \lambda^{-1} x^T(n) P(n - 1) x(n)} \]

Using this definition

\[ P(n) = \lambda^{-1} P(n - 1) - \lambda^{-1} k(n) x^T(n) P(n - 1) \]

\( k(n) \) can be written as

\[ k(n) = \left[ \lambda^{-1} P(n - 1) - \lambda^{-1} k(n) x^T(n) P(n - 1) \right] x(n) \]

\[ = \hat{R}^{-1}(n) x(n) \]

\( \Rightarrow k(n) \) is the repres. of \( x(n) \) on the column space of \( \hat{R}(n) \)
RLS – Recursive Weight Update

- $\mathbf{w}(n+1) = \mathbf{w}^o(n)$
- Using

$$
\mathbf{w}(n+1) = \mathbf{R}^{-1}(n)\hat{\mathbf{p}}(n) = \mathbf{P}(n)\hat{\mathbf{p}}(n)
$$

$$
\hat{\mathbf{p}}(n) = \lambda \hat{\mathbf{p}}(n-1) + \mathbf{x}(n)d(n)
$$

$$
\mathbf{P}(n) = \lambda^{-1} \mathbf{P}(n-1) - \lambda^{-1} \mathbf{k}(n)\mathbf{x}^T(n)\mathbf{P}(n-1)
$$

$$
\mathbf{k}(n) = \mathbf{P}(n)\mathbf{x}(n)
$$

$$
e(n) = d(n) - \mathbf{x}^T(n)\mathbf{w}(n)
$$

yields

\[
\mathbf{w}(n+1) = \mathbf{w}(n) + \mathbf{k}(n)e(n)
\]

RLS weight update
The RLS convergence speed is not affected by the eigenvalues of $\hat{R}(n)$.

Initialization: $\hat{p}(0) = 0$, $\hat{R}(0) = \delta I$, $\delta \simeq (1 - \lambda)\sigma_x^2$.

This results in $\hat{R}(n)$ changed to $\lambda^n \delta I + \hat{R}(n)$

$\Rightarrow$ Biased estimate of $\omega^o(n)$ for small $n$

(No problem for $\lambda < 1$ and large $n$)

Numerical problems in finite precision

$\triangleright$ Unstable for $\lambda = 1$

$\triangleright$ Loss of symmetry in $P(n)$

$\star$ Evaluate only upper (lower) part and diagonal of $P(n)$

$\star$ Replace $P(n)$ with $[P(n) + P^T(n)]/2$ after updating from $P(n - 1)$

Numerical problems when $x(n) \to 0$ and $\lambda < 1$
Some Research Results


Performance Comparison

- System identification, N=100
- Input – AR(1) \( x(n) = 0.9x(n - 1) + v(n) \)
- AP algorithm with \( P = 2 \)
- Step sizes designed for equal LMS and NLMS performances
- Impulse response to be estimated
Excess Mean Square Error - LMS, NLMS and AP

EMSE for LMS, NLMS and AP(2)

EMSE (dB) vs iteration

José Bermudez (UFSC)
Adaptive Filtering
IRIT - Toulouse, 2011
Excess Mean Square Error - LMS, NLMS, AP and RLS
Performance Comparisons
Computational Complexity

- Adaptive filter with $N$ real coefficients and real signals
- For the AP algorithm, $K = P + 1$

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>×</th>
<th>+</th>
<th>\slash</th>
<th>\approx \text{factor}</th>
</tr>
</thead>
<tbody>
<tr>
<td>LMS</td>
<td>$2N + 1$</td>
<td>$2N$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>NLMS</td>
<td>$3N + 1$</td>
<td>$3N$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>AP</td>
<td>$(K^2 + 2K)N + K^3 + K$</td>
<td>$(K^2 + 2K)N + K^3 + K^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>RLS</td>
<td>$N^2 + 5N + 1$</td>
<td>$N^2 + 3N$</td>
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<td>1</td>
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</table>

For $N = 100, P = 2$

<table>
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<tr>
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<th>×</th>
<th>+</th>
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<th>\approx \text{factor}</th>
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<td>200</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>NLMS</td>
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</tr>
<tr>
<td>AP</td>
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<td>1,536</td>
<td></td>
<td>7.5</td>
</tr>
<tr>
<td>RLS</td>
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<td>10,300</td>
<td>1</td>
<td>52.5</td>
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</table>
Typical values for acoustic echo cancellation ($N = 1024, P = 2$)

<table>
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<th>$+$</th>
<th>/</th>
<th>$\simeq$ factor</th>
</tr>
</thead>
<tbody>
<tr>
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<td>2,048</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>NLMS</td>
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<td>3,072</td>
<td>1</td>
<td>1.5</td>
</tr>
<tr>
<td>AP</td>
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<td>15,396</td>
<td></td>
<td>7.5</td>
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<tr>
<td>RLS</td>
<td>1,053,697</td>
<td>1,051,648</td>
<td>1</td>
<td>514</td>
</tr>
</tbody>
</table>
How to Deal with Computational Complexity?

- Not an easy task!!!
- There are "fast" versions for some algorithms (especially RLS)
- What is usually not said is that ... speed can bring
  - Instability
  - Increased need for memory

⇒ *Most applications rely on simple solutions*
Understanding the Adaptive Filter Behavior
What are Adaptive Filters?

Adaptive filters are, by design, systems that are

- Time-variant ($w(n)$ is time-variant)
- Nonlinear ($y(n)$ is a nonlinear function of $x(n)$)
- Stochastic ($w(n)$ is random)

**LMS:**

$$w(n + 1) = w(n) + \mu e(n)x(n)$$

$$= [I - \mu x(n)x^T(n)] x(n) + \mu d(n)x(n)$$

$$y(n) = x^T(n)w(n)$$

- They are difficult to understand
- Different applications and signals require different analyses
- Simplifying assumptions are necessary

**Good design requires good analytical models.**
LMS Analysis

Basic equations:

\[ d(n) = x^T(n)w^o + z(n) \]

\[ e(n) = d(n) - x^T(n)w(n) \]

\[ w(n + 1) = w(n) + \mu e(n)x(n) \]

\[ v(n) = w(n) - w^o \quad \text{(study about the optimum weight vector)} \]

Mean Weight Behavior:

*Neglecting dependence between \( v(n) \) and \( x(n)x^T(n) \),*

\[ E[v(n + 1)] = (I - \mu R_{xx})E[v(n)] \]

- Follows the steepest descent trajectory
- Convergence condition: \( 0 < \mu < 2/\lambda_{\text{max}} \)
- \( \lim_{n \to \infty} E[w(n)] = w^o \)

However: \( w(n) \) is random

\( \Rightarrow \) *We need to study its fluctuations about* \( E[w(n)] \to \text{MSE} \)
The Mean-Square Estimation Error

\[ e(n) = e_o(n) - x^T(n)v(n), \quad e_o(n) = d(n) - x^T(n)w^o \]

- Square the error equation
- Take the expected value
- Neglect the statistical dep. between \( v(n) \) and \( x(n)x^T(n) \)

\[ J_{ms}(n) = E[e_o^2(n)] + Tr\{R_{xx}K(n)\}, \quad K(n) = E[v(n)v^T(n)] \]

- This expression is independent of the adaptive algorithm
- Effect of the alg. on \( J_{ms}(n) \) is determined by \( K(n) \)
- \( J_{ms_{min}} = E[e_o^2(n)] \)
- \( J_{ms_{ex}} = Tr\{R_{xx}K(n)\} \)
The Behavior of $K(n)$ for LMS

- Using the basic equations

$$v(n + 1) = \left[ I + \mu x(n)x^T(n) \right] v(n) + \mu e_o(n)x(n)$$

- Post-multiply by $v^T(n + 1)$
- Take the expected value assuming $v(n)$ and $x(n)x^T(n)$ independent
- Evaluate the expectations
  - Higher order moments of $x(n)$ require input pdf
- Assuming Gaussian inputs

$$K(n + 1) = K(n) - \mu \left[ R_{xx}K(n) + K(n)R_{xx} \right]$$
$$+ \mu^2 \left\{ R_{xx} \text{Tr}[R_{xx}K(n)] + 2R_{xx}K(n)R_{xx} \right\}$$
$$+ \mu^2 R_{xx}J_{ms_{\text{min}}}$$
Design Guidelines Extrated from the Model

- **Stability**

\[ 0 < \mu < \frac{m}{\text{Tr}[R_{xx}]} ; \quad m = 2 \text{ or } m = \frac{2}{3} \]

- **\( J_{ms}(n) \)** converges as a function of

\[ [(1 - \mu \lambda_i)^2 + 2\mu^2 \lambda_i^2]^n , \quad i = 1, \ldots, N \]

- **Steady-State**

\[ J_{ms_{\text{ex}}} (\infty) \simeq \frac{\mu}{2} \text{Tr}[R_{xx}] J_{ms_{\text{min}}} \]

\[ \mathcal{M}(\infty) = \frac{J_{ms_{\text{ex}}} (\infty)}{J_{ms_{\text{min}}}} \quad \text{(MSE Misadjustment)} \]
Model Evaluation

\[ x(n) = \alpha x(n - 1) + v(n) \]

- **Input:** White noise
- **Linear system identification** - FIR with 100 coefficients
- **Step-size** \( \mu = 0.05 \)
- **Noise power** \( \sigma_v^2 = 10^{-6} \)

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Input: AR(1), \( \alpha = 0.8 \)
Merci pour votre attention!!!